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AN IDENTITY FOR A CLASS OF ARITHMETICAL FUNCTIONS OF SEVERAL VARIABLES

PENTTI HAUKKANEN

Department of Mathematical Sciences
University of Tempere
P.O. Box 607
SF-33101 Tampere
FINLAND

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ABSTRACT. Johnson [1] evaluated the sum $\sum_{d \mid n} |C(d;r)|$, where $C(n;r)$ denotes Ramanujan's trigonometric sum. This evaluation has been generalized to a wide class of arithmetical functions of two variables. In this paper, we generalize this evaluation to a wide class of arithmetical functions of several variables and deduce as special cases the previous evaluations.

KEY WORDS AND PHRASES. Arithmetical functions of several variables, multiplicative functions, Ramanujan's sum and its generalizations.
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1. INTRODUCTION.

In [1], Johnson evaluated the sum

$$\sum_{d \mid n} |C(d;r)|,$$

where $C(n;r)$ denotes Ramanujan's trigonometric sum. This evaluation has been generalized by Chidambaraswamy and Krishnaiah [2], Johnson [3], and Redmond [4]. The generalization given by Chidambaraswamy and Krishnaiah is the most extensive one and contains the other evaluations as special cases. They evaluated the sum

$$\sum_{d^k \mid n} |S^{(k)}(d^k;r)|,$$

where k is a positive integer and

$$S^{(k)}(n;r) = S^{(k)}_{g,h}(n;r) = \sum_{d^k \mid (n,r^k)_k} g(d)\mu(r/d)h(r/d),$$

g and h being given arithmetical functions, μ being the well-known Möbius function and $(x,y)_k$ standing for the greatest common kth power divisor of x and y.

In this paper, we shall evaluate the more extensive sum

$$\sum_{d_1^k \mid n_1} \cdots \sum_{d_j^k \mid n_j} |S^{(k)}(d_1^k, \ldots, d_j^k, n_{j+1}^k, \ldots, n_u^k; r)|,$$

where

$$S^{(k)}(n_1,\dots,n_u;r) = \sum_{d^k \mid ((n_i),r^k)_k} g(d)\mu(r/d)h(r/d).$$

Here $(n_i) = (n_1,\dots,n_u)$, the greatest common divisor of $n_1,\dots,n_u$.

2. RESULTS.

For a positive integer k let τ_k denote the arithmetical function such that $\tau_k(n)$ is the number of positive kth power divisors of n.

For a given $(u+1)$-tuple $n_1,\dots,n_u,r$ of positive integers let $\widehat{r}$ denote the largest divisor of r such that $(\widehat{r},n_i)=1$ for all $i=1,\dots,u$. Also for each $i=1,\dots,u$ let $\widehat{n}_i$ denote the largest divisor of n_i such that $(\widehat{n}_i,r)=1$. We write $\tilde{r}$ for $r/\widehat{r}$ and $\tilde{n}_i$ for $n_i/\widehat{n}_i$. The symbol r_* denotes the quotient of r by its largest squarefree divisor.

Let $n_i = \prod_p p^{a_i}(a_i = a_i(p)), r = \prod_p p^b(b=b(p))$ be the canonical decompositions of $n_i (i=1,\dots,u)$ and r. When $r_*^k \mid n_i$, let $c_i (c_i = c_i(p,k))$ be determined so that $p^{kc_i} \mid n_i/r_*^k$ and $p^{k(c_i+1)} \nmid n_i/r_*^k$; that is, $c_i = [a_i/k]-b+1$ if $b \geq 1$, and $c_i = [a_i/k]$ if $b=0$.

THEOREM. If g is a completely multiplicative function, h a multiplicative function and $1 \leq j \leq u$, then

$$\begin{aligned}
&\sum_{d_1^k \mid n_1} \cdots \sum_{d_j^k \mid n_j} |S^{(k)}(d_1^k,\dots,d_j^k,n_{j+1}^k,\dots,n_u^k;r)| \\
&= \tau_k(\widehat{n}_1)\cdots\tau_k(\widehat{n}_j)\,|g(r_*)| \\
&\times \prod_{\substack{p \mid r \\ b \leq a}} \Big\{\{(c_1+1)\cdots(c_j+1)-c_1\cdots c_j\}\,|h(p)| + c_1\cdots c_j\,|g(p)-h(p)|\Big\} \\
&\times \prod_{\substack{p \mid r \\ b > a}} (c_1+1)\cdots(c_j+1)\,|h(p)|
\end{aligned} \tag{2.1}$$

or 0 according as $r_*^k \mid (n_1,\dots,n_j,n_{j+1}^k,\dots,n_u^k)$ or not, where $a = min\{a_{j+1},\dots,a_u\}$. (If $j=u$, we put $a=\infty$.)

PROOF. Let $r_*^k \nmid (n_1,\dots,n_j,n_{j+1}^k,\dots,n_u^k)$. Suppose $d_i^k \mid n_i$ for each $i=1,\dots,j$. Write

$$S^{(k)}(d_1^k,\dots,d_j^k,n_{j+1}^k,\dots,n_u^k;r) = \sum_{\substack{\delta \mid r \\ \delta \mid d_1,\dots,d_j,n_{j+1},\dots,n_u}} g(\delta)\mu(r/\delta)h(r/\delta)$$

Here $r_* \nmid (d_1,\dots,d_j,n_{j+1},\dots,n_u)$ and so $\mu(r/\delta)=0$ for all δ in the sum. Thus the left-hand side of (2.1) is equal to 0.

Let $r_*^k \mid (n_1,\dots,n_j,n_{j+1}^k,\dots,n_u^k)$. Suppose $d_i^k \mid n_i$ for each $i=1,\dots,j$. Let $\widehat{d}_i$ and $\tilde{d}_i$ be defined in a similar way to $\widehat{n}_i$ and $\tilde{n}_i$. Then the multiplicativity of $S^{(k)}(n_1,\dots,n_u;r)$ in the variables $n_1,\dots,n_u,r$ implies

$$\begin{aligned}
&S^{(k)}(d_1^k,\dots,d_j^k,n_{j+1}^k,\dots,n_u^k;r) \\
&\quad = S^{(k)}(\tilde{d}_1^k\ \widehat{d}_1^k,\dots,\tilde{d}_j^k\ \widehat{d}_j^k,\ \tilde{n}_{j+1}^k\ \widehat{n}_{j+1}^k,\dots,\tilde{n}_u^k\ \widehat{n}_u^k;\ \tilde{r}\ \widehat{r}) \\
&\quad = S^{(k)}(\tilde{d}_1^k\ \widehat{d}_1^k,\dots,\tilde{d}_j^k,\ \tilde{n}_{j+1}^k,\dots,\tilde{n}_u^k;\ \tilde{r})S^{(k)}(\widehat{d}_1^k,\dots,\widehat{d}_j^k,\ \widehat{n}_{j+1}^k,\dots,\widehat{n}_u^k;\ \widehat{r}) \\
&\quad = S^{(k)}(\tilde{d}_1^k,\dots,\tilde{d}_j^k,\ \tilde{n}_{j+1}^k,\dots,\tilde{n}_u^k;\ \tilde{r})S^{(k)}(1;\widehat{r})S^{(k)}(\widehat{d}_1^k,\dots,\widehat{d}_j^k,\ \widehat{n}_{j+1}^k,\dots,\widehat{n}_u^k;\ 1) \\
&\quad = S^{(k)}(\tilde{d}_1^k,\dots,\tilde{d}_j^k,\ \tilde{n}_{j+1}^k,\dots,\tilde{n}_u^k;\ \tilde{r})\mu(\widehat{r})h(\widehat{r}).
\end{aligned}$$

Thus, denoting by L the left-hand side of (1.1), we obtain

$$L = |h(\tilde{r})| \sum_{d_1^k | n_1} \cdots \sum_{d_j^k | n_j} |S^{(k)}(\tilde{d}_1^k, \ldots, \tilde{d}_j^k, \tilde{n}_{j+1}^k, \ldots, \tilde{n}_u^k; \tilde{r})|$$

$$= |h(\tilde{r})| \sum_{\delta_1^k | \tilde{n}_1} \cdots \sum_{\delta_j^k | \tilde{n}_j} |S^{(k)}(\delta_1^k, \ldots, \delta_j^k, \tilde{n}_{j+1}^k, \ldots, \tilde{n}_u^k; \tilde{r})| \sum_{e_1^k | \hat{n}_1} \cdots \sum_{e_j^k | \hat{n}_j} 1.$$

The sum over $e_1, \ldots, e_j$ is equal to $\tau_k(\hat{n}_1) \cdots \tau_k(\hat{n}_j)$.

By the multiplicativity of the function $S^{(k)}(n_1, \ldots, n_u; r)$ and the properties of the Mōbius function μ, we have

$$\sum_{\delta_1^k | \tilde{n}_1} \cdots \sum_{\delta_j^k | \tilde{n}_j} |S^{(k)}(\delta_1^k, \ldots, \delta_j^k, \tilde{n}_{j+1}^k, \ldots, \tilde{n}_u^k; \tilde{r})|$$

$$= \prod_{p | \tilde{r}} \sum_{i_1 = 0}^{[a_1/k]} \cdots \sum_{i_j = 0}^{[a_j/k]} |S^{(k)}(p^{i_1 k}, \ldots, p^{i_j k}, p^{a_{j+1} k}, \ldots, p^{a_u k}; p^b)|$$

$$= \prod_{p | \tilde{r}} \sum_{i_1 = b-1}^{[a_1/k]} \cdots \sum_{i_j = b-1}^{[a_j/k]} \Big| \sum_{\substack{d | p^b \\ d | p^{i_1}, \ldots, p^{i_j}, p^{a_{j+1}}, \ldots p^{a_u}}} g(d)(\mu h)(p^b/d) \Big|$$

$$= \prod_{\substack{p | \tilde{r} \\ b \le a}} \left\{ \{(c_1+1) \cdots (c_j+1) - c_1 \cdots c_j\} |g(p^{b-1}) h(p)| + c_1 \cdots c_j |g(p^{b-1})| |g(p) - h(p)| \right\}$$

$$\times \prod_{\substack{p | \tilde{r} \\ b > a}} (c_1+1) \cdots (c_j+1) |g(p^{b-1}) h(p)|.$$

Thus

$$L = \tau_k(\hat{n}_1) \cdots \tau_k(\hat{n}_j) |g(r_*)| |h(\hat{r})|$$

$$\times \prod_{\substack{p | \tilde{r} \\ b \le a}} \left\{ \{(c_1+1) \cdots (c_j+1) - c_1 \cdots c_j\} |h(p)| + c_1 \cdots c_j |g(p) - h(p)| \right\}$$

$$\times \prod_{\substack{p | \tilde{r} \\ b > a}} (c_1+1) \cdots (c_j+1) |h(p)|.$$

If $p | \hat{r}$, then $b = 1$ and $c_1 = \cdots = c_j = a = 0$. We thus arrive at our result.

EXAMPLES. If $j = u = 1$ in the Theorem, we obtain the result given in [2]; that is,

$$\sum_{d_1^k | n_1} |S^{(k)}(d_1; r)| = \tau_k(\hat{n}_1) |g(r_*)| \prod_{p | r} (|h(p)| + c_1 |g(p) - h(p)|) \tag{2.2}$$

or 0 according as $r_*^k | n_1$ or not. For special cases of (2.2) we refer to [2]. If $g(n) = n^{ku}$ and $h(n) = 1$ for all $n \in \mathbb{N}$, then the function $S^{(k)}(n_1, \ldots, n_u; r)$ reduces to the generalized Ramanujan's sum given in [5]. If in addition, $k = 1$, then we obtain the generalized Ramanujan's sum given in [6]. Thus the Theorem could be specialized to those functions, too.

REFERENCES

1. JOHNSON, K.R., A result for the 'other' variable of Ramanujan's sum, *El. Math.* 38 (1983), 122-124.
2. CHIDAMBARASWAMY, J. and KRISHNAIAH, P.V., An identity for a class of arithmetical functions of two variables, *Internat. J. Math. Math. Sci.* 11 (1988), 351-354.
3. JOHNSON, K.R., An explicit formula for sums of Ramanujan type sums, *Indian J. Pure Appl. Math* 18 (1987), 675-677.
4. REDMOND, D., A generalization of a result of K.R. Johnson, *Tsukuba J. Math.* 13 (1989), 99-105.
5. SURYANARAYANA, D. and WALKER, D.T., Some generalizations of an identity of Subhankulov, *Canad. Math. Bull.* 20 (1977), 489-494.
6. COHEN, E., A class of arithmetical functions in several variables with applications to congruences, *Trans. Amer. Math. Sci.* 96 (1960), 355-381.